Dense Coding and Teleportation.

These are two simple communication but important protocols that can be used as primitives for more complicated tasks. They are based on entangled states of the Bell-type (or EPR pairs of qubits).

<u>**Definition**</u>: Let $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ the Hilbert space of a composite system which is "bipartite". <u>Product states</u> $|\psi\rangle \in \mathcal{H}$ are those states that can be written in the form

$$|\psi\rangle = |\varphi\rangle_A \otimes |\chi\rangle_B$$

for some $|\varphi\rangle_A \in \mathcal{H}_A$, $|\chi\rangle_B \in \mathcal{H}_B$.

<u>Entangled states</u> are those that cannot be written in this form.

Examples: $\mathcal{H}_A = \mathbb{C}^2$, $\mathcal{H}_B = \mathbb{C}^2$

- $|0\rangle \otimes |0\rangle$, $|0\rangle \otimes |1\rangle$, $|1\rangle \otimes |0\rangle$, $|1\rangle \otimes |1\rangle$

are product states (of the computational basis of $\mathbb{C}^2 \otimes \mathbb{C}^2$).

- $\frac{1}{2}\Big(|00\rangle + |01\rangle + |10\rangle + |11\rangle\Big)$ is a product state

because it is "factorized" as $\dfrac{|0\rangle + |1\rangle}{\sqrt{2}} \otimes \dfrac{|0\rangle + |1\rangle}{\sqrt{2}}$.

- Same for

$$\frac{1}{2}\Big(|00\rangle - |01\rangle - |10\rangle + |11\rangle\Big) = \frac{|0\rangle + |1\rangle}{\sqrt{2}} \otimes \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

- But $\frac{1}{2}\Big(|00\rangle + |01\rangle - |10\rangle + |11\rangle\Big)$ is

entangled. Indeed suppose we could write it as

$(\alpha|0\rangle + \beta|1\rangle) \otimes (\gamma|0\rangle + \delta|1\rangle)$ then we should have

$$\alpha\gamma = \frac{1}{2}, \quad \alpha\delta = \frac{1}{2}, \quad \beta\gamma = -\frac{1}{2}, \quad \beta\delta = \frac{1}{2}$$

but this implies $\alpha\gamma\,\alpha\delta\,\beta\delta = \frac{1}{8} \Leftrightarrow \alpha^2\delta^2\beta\gamma = \frac{1}{8}$

$\Rightarrow \beta\gamma > 0$. But $\beta\gamma = -\frac{1}{2} < 0$

hence a contradiction!

Remark: from the last two examples it is apparent that for N qubits it might not even be easy to decide if a state is product or entangled ...

The Bell states.

These are basic because (in many ways) they are the "most" entangled states of two qubits. (Here we do not quantify entanglement however/ this is another interesting topic). There are four Bell states:

$$
\begin{cases}
|\psi^{\pm}\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle \pm |11\rangle\right) \\
|\phi^{\pm}\rangle = \frac{1}{\sqrt{2}}\left(|01\rangle \pm |10\rangle\right)
\end{cases}
$$

The four Bell states form an orthonormal basis of $\mathbb{C}^2 \otimes \mathbb{C}^2$.

__Exercise.__ Let
$$\begin{cases} |\vartheta\rangle = \cos\vartheta\,|0\rangle + \sin\vartheta\,|1\rangle \\ |\vartheta_\perp\rangle = \sin\vartheta\,|0\rangle - \cos\vartheta\,|1\rangle \end{cases}$$

Show that
$$|\psi^+\rangle = \frac{1}{\sqrt{2}}\left(|\vartheta\rangle \otimes |\vartheta\rangle + |\vartheta_\perp\rangle \otimes |\vartheta_\perp\rangle\right).$$

So in a sense $|\psi^+\rangle$ is "completely isotropic".

The situation is similar for the other states.

__Measurements.__ (for $|\psi^+\rangle$)

- A computational basis measurement leaves us with state $|0\rangle \otimes |0\rangle$ or $|1\rangle \otimes |1\rangle$ with prob $1/2$. So measurement results are correlated.
- If only A measures (say) $|\psi^+\rangle$ would be projected on
$$\begin{cases} (|0\rangle\langle 0| \otimes \mathbb{1})\,|\psi^+\rangle = \frac{1}{2}\,|0\rangle \otimes |0\rangle \\ (|1\rangle\langle 1| \otimes \mathbb{1})\,|\psi^+\rangle = \frac{1}{2}\,|1\rangle \otimes |1\rangle \end{cases}$$

So from Alice's point of view Bob's state is $|0\rangle$ or $|1\rangle$. However Bob cannot know that A has made a measurement until she communicates the result. This means that if A & B <u>dont communicate</u> Bob should get the same statistics wether he measures (in same basis) before or after Alice. Let us check!

→ <u>Meas before Alice</u> in $\{|\theta\rangle, |\theta_\perp\rangle\}$ basis of Bob.

Outcome is $|\theta\rangle_B$ or $|\theta_\perp\rangle_B$ and since

$$|\psi^+\rangle = \frac{1}{\sqrt{2}}\left(|\theta\rangle_A \otimes |\theta\rangle_B + |\theta_\perp\rangle_A \otimes |\theta_\perp\rangle_B\right)$$

Bob has the statistics $\text{prob}(\theta) = \frac{1}{2}$, $\text{prob}(\theta_\perp) = \frac{1}{2}$.

→ <u>Meas after Alice</u> in $\{|\theta\rangle, |\theta_\perp\rangle\}$ basis of Bob.

Alice measured in $\{|0\rangle, |1\rangle\}$ before Bob and got $|0\rangle$ with prob $1/2$ and $|1\rangle$ with prob $1/2$.

So Bob will anyway get $|\vartheta\rangle$ or $|\vartheta_\perp\rangle$, and his statistics is

$$\text{prob}(\vartheta) = \text{prob}(\vartheta / \underbrace{A \text{ got } |0\rangle}_{\substack{\text{state after A's meas} \\ |0\rangle_A \otimes |0\rangle_B}}) \cdot \frac{1}{2} + \text{prob}(\vartheta / \underbrace{A \text{ got } |1\rangle}_{\substack{\text{state after A's meas} \\ |1\rangle_A \otimes |1\rangle_B}}) \cdot \frac{1}{2}$$

prob for B is $|\langle\vartheta|0\rangle_B|^2$
$= (\cos\vartheta)^2$

prob for B is
$|\langle\vartheta|1\rangle_B|^2$
$= (\sin\vartheta)^2$

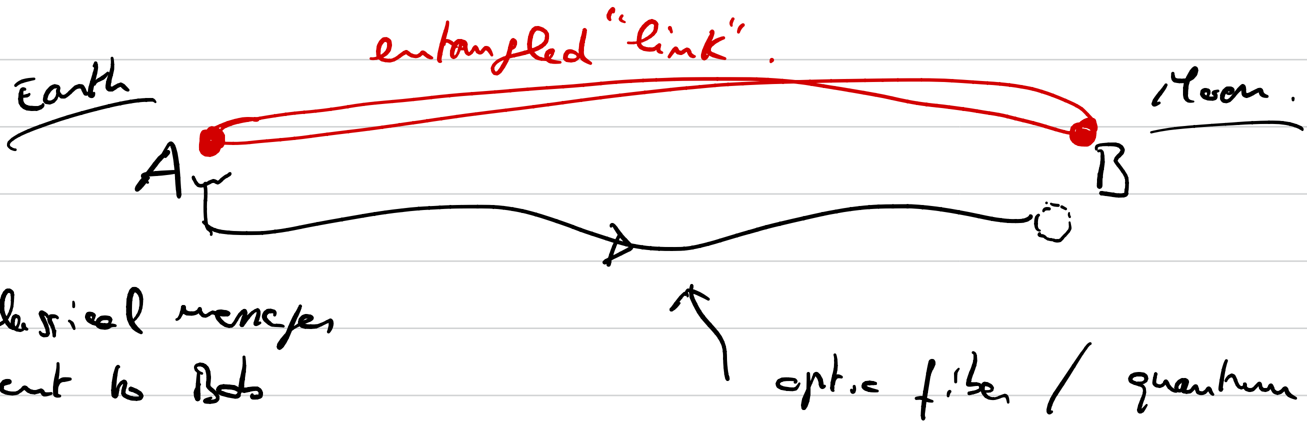
$$\Rightarrow \quad \text{prob}(\vartheta) = (\cos\vartheta)^2 \cdot \frac{1}{2} + (\sin\vartheta)^2 \cdot \frac{1}{2}$$

$$= \frac{1}{2}.$$

$\blacksquare$

Dense Coding Protocol:

In this protocol: Alice transfers 2 classical bits of information to Bob by physically sending 1 qu-bit over a quantum channel.



classical messages sent to Bob
00
01
10
11

optic fiber / quantum communication channel.
used only once.

To achieve this A & B share an entangled pair $|\psi^+\rangle$. This form of communication is called "entanglement-assisted communication". Without the entangled link one would be able to send only 1 bit through one use of the channel. With the entangled link this capacity is doubled (here channels are not noisy).

Protocol.

It is assume that A & B have established an entangled link, i.e, an EPR pair in state $|\psi^+\rangle$.

- to send message 00 : Alice sends her share of $|\psi^+\rangle$.
- to send message 01 : Alice applies the unitary operator $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ to her share of $|\psi^+\rangle$
- to send message 10 : Alice applies the unitary op $Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ to her share of $|\psi^+\rangle$.
- to send message 11 : Alice applies the unitary op $ZX = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ to her share of $|\psi^+\rangle$

Bob the qubit (A's share of $|\psi^+\rangle$). He is now in possession of the full state in his lab :

- $\mathbb{1}_A \otimes \mathbb{1}_B |\psi^+\rangle = |\psi^+\rangle$

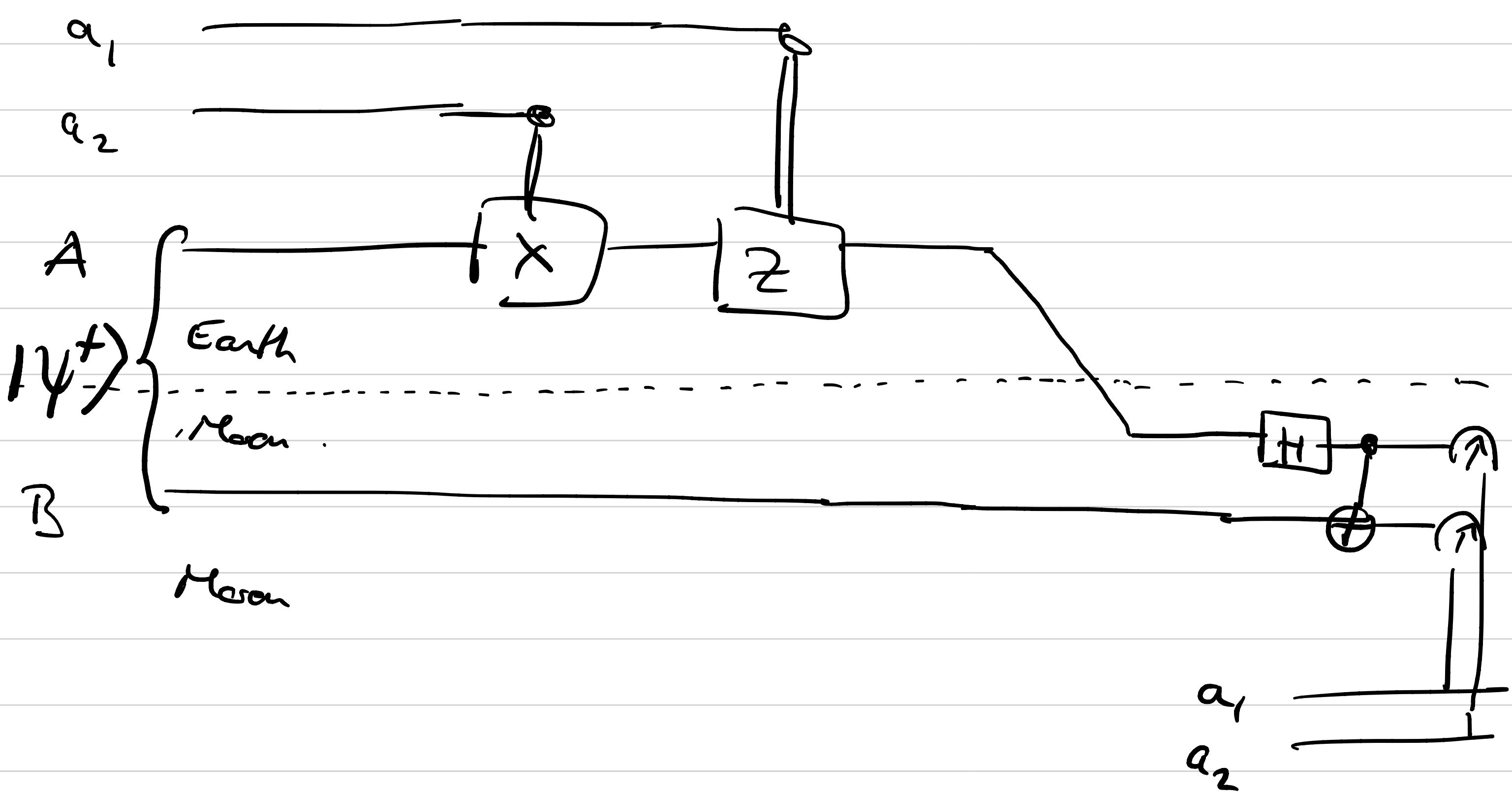
- $X_A \otimes \mathbb{1}_B |\psi^+\rangle = \frac{1}{\sqrt{2}}\left(|10\rangle + |01\rangle\right) = |\phi^+\rangle$

- $Z_A \otimes \mathbb{1}_B |\psi^+\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle - |11\rangle\right) = |\psi^-\rangle$

- $Z_A X_A \otimes \mathbb{1}_B |\psi^+\rangle = \frac{1}{\sqrt{2}}\left(-|10\rangle + |01\rangle\right) = |\phi^-\rangle$

Finally Bob makes a <u>Bell basis measurement</u> to decide which is the state in his possession. If his meas yields $|\psi^+\rangle$ he records (00), if his meas yields $|\phi^+\rangle$ he records (01), if his meas yields $|\psi^-\rangle$ he records (10), if his meas yields $|\phi^-\rangle$ he records (11).

A possible circuit representation of the dense coding protocol.



This protocol is often summerized as

$$\underbrace{2 \text{ c-bits}}_{\text{the classical message}} = \underbrace{1 \text{ qu-bit}}_{\text{the qubit sent}} + \underbrace{1 \text{ e-bit}}_{\text{the EPR pair}}.$$

But note that there are 2 qubits involved in total.

Teleportation Protocol.

In this protocol: Alice wants to "transfer" (we say "teleport") a qu-bit to Bob by physically sending two classical bits (and not physically sending the qubit).

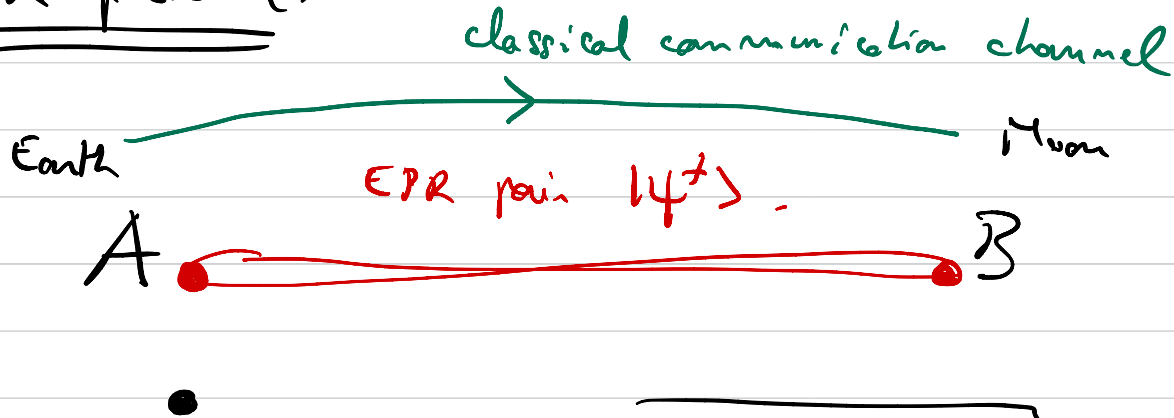
To achieve this A & B preshare an entangled pair as before,

Remarks:

1) This $\overbrace{\text{is}}^{\text{task}}$ somehow dual to the previous dense coding task. However this time there will be 3 qubits involved in total.

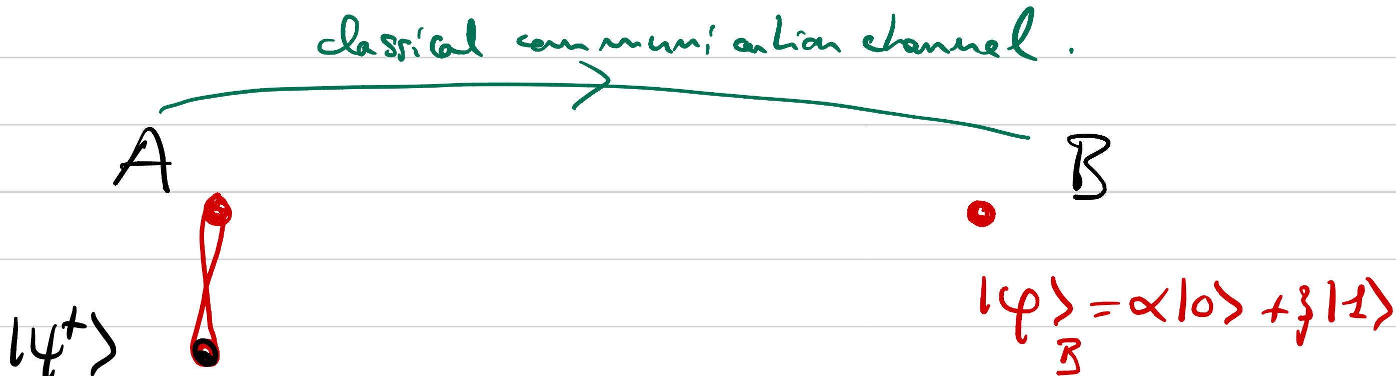
2) The qubit state is teleported to Bob and is lost on Alice's side. Otherwise the no-cloning thm would be violated.

The protocol:



$$|\varphi\rangle_A = \alpha|0\rangle + \beta|1\rangle$$

initial situation



final situation

- Alice makes a measurement in her Bell basis

$$\{ |\varphi^+\rangle_A, |\phi^+\rangle_A, |\psi^-\rangle_A, |\phi^-\rangle_A \}$$

and records (00) (01) (10) (11).

- She sends the 2-classical bits message to Bob.

- Bob receives 2 classical bits and does unitary transformations accordingly to his share of $|\psi^+\rangle$

→ if he receives (00) he applies $\mathbb{1}_B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

→ if he receives (01) he applies $X_B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

→ if he receives (10) he applies $Z_B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

→ if he receives (11) he applies $Z_B X_B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

We claim that this garantees he has $|\varphi_B\rangle$.

Note that he never measures or even never attempts to "know" what $|\varphi_B\rangle$ is.

Analysis of teleportation protocol:

The global initial state lives in the Hilbert space $\mathbb{C}^2_{A_1} \otimes \mathbb{C}^2_B \otimes \mathbb{C}^2_{A_2}$;

$$|\psi^+\rangle_{A_1B} \otimes |\varphi\rangle_{A_2} = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right) \otimes \left(\alpha|0\rangle + \beta|1\rangle\right)$$

$$= \frac{\alpha}{\sqrt{2}}\,|0_{A_1}\,0_B\,0_{A_2}\rangle + \frac{\beta}{\sqrt{2}}\,|0_{A_1}\,0_B\,1_{A_2}\rangle$$

$$+ \frac{\alpha}{\sqrt{2}}\,|1_{A_1}\,1_B\,0_{A_2}\rangle + \frac{\beta}{\sqrt{2}}\,|1_{A_1}\,1_B\,1_{A_2}\rangle$$

After the measurement of Alice we have four possible outcomes in her lab $|\psi^+\rangle_{A_1A_2}$, $|\phi^+\rangle_{A_1A_2}$, $|\psi^-\rangle_{A_1A_2}$, $|\phi^-\rangle_{A_1A_2}$. To get the global state after Alice's measurement we compute (here for the first possibility):

$$\overbrace{\left(|\psi^+\rangle_{A_1A_2}\langle\psi^+|_{A_1A_2}\otimes \mathbb{1}_B\right)}^{\text{projector of measurement.}} |\psi^+\rangle_{A_1B}\otimes|\varphi\rangle_{A_2}$$

$$= |\psi^+\rangle_{A_1A_2}\otimes \underbrace{\frac{1}{\sqrt{2}}\left(\langle 0_{A_1}\, 0_{A_2}| + \langle 1_{A_1}\, 1_{A_2}|\right)\left(\cdots\cdots\right)}_{\frac{1}{\sqrt{2}}\left\{\frac{\alpha}{\sqrt{2}}|0_B\rangle+\frac{\beta}{\sqrt{2}}|1_B\rangle\right\}}$$

$$= \frac{1}{2}\, |\psi^+\rangle_{A_1A_2}\otimes \underbrace{\left(\alpha|0\rangle_B+\beta|1\rangle_B\right)}_{|\varphi\rangle_B}.$$

This factor $1/2$ is divided out when we normalize the state,

$\boxed{\text{Similarly}}$:

$$\left(|\phi^+\rangle_{A_1A_2}\overbrace{\langle\phi^+|_{A_1A_2}}^{\frac{1}{\sqrt{2}}\left(\langle 0_{A_1} 1_{A_2}| + \langle 1_{A_1} 0_{A_2}|\right)}\otimes \mathbb{1}_B\right) |\psi^+\rangle_{A_1B}\otimes|\varphi\rangle_{A_2}$$

$$= |\phi^+\rangle_{A_1A_2}\otimes \frac{1}{\sqrt{2}}\left\{\beta|0\rangle_B+\alpha|1\rangle_B\right\}$$

- $$\Big(|\psi^-\rangle_{A_1A_2}\overbrace{\langle\psi^-|_{A_1A_2}}^{\frac{1}{\sqrt{2}}\left(\langle 00_{A_1A_2}| - \langle 11_{A_1A_2}|\right)}\otimes \mathbb{1}_B\Big)|\psi^+\rangle_{A_1B}\otimes|\varphi\rangle_{A_2}$$

$$= |\psi^-\rangle_{A_1A_2}\otimes\frac{1}{\sqrt{2}}\left\{\alpha|0\rangle_B - \beta|1\rangle_B\right\}.$$

- $$\Big(|\phi^-\rangle_{A_1A_2}\langle\phi^-|_{A_1A_2}\otimes \mathbb{1}_B\Big)|\psi^+\rangle_{A_1B}\otimes|\varphi\rangle_{A_2}$$

$$= |\phi^-\rangle_{A_1A_2}\otimes\frac{1}{\sqrt{2}}\left\{\beta|0\rangle_B - \alpha|1\rangle_B\right\}$$

In summary:

Alice has	Bob has
$\|\psi^+\rangle_{A_1A_2}$	$\alpha\|0\rangle_B + \beta\|1\rangle_B$
$\|\phi^+\rangle_{A_1A_2}$	$\beta\|0\rangle_B + \alpha\|1\rangle_B$
$\|\psi^-\rangle_{A_1A_2}$	$\alpha\|0\rangle_B - \beta\|1\rangle_B$
$\|\phi^-\rangle_{A_1A_2}$	$\beta\|0\rangle_B - \alpha\|1\rangle_B$.

Classical communication phase

$|\psi^+\rangle_{A_1A_2}$ Alice sends 00 $\Rightarrow$ Bob applies $\mathbb{1}_B(\alpha|0\rangle+\beta|1\rangle)$

$= |\varphi_B\rangle$. ✓

$|\phi^+\rangle_{A_1A_2}$ Alice sends 01 $\Rightarrow$ Bob applies $X_B(\beta|0\rangle+\alpha|1\rangle)$

$= \beta|1\rangle+\alpha|0\rangle$

$= |\varphi_B\rangle$ ✓

$|\psi^-\rangle_{A_1A_2}$ Alice sends 10 $\Rightarrow$ Bob applies $Z_B(\alpha|0\rangle-\beta|1\rangle)$

$= \alpha|0\rangle+\beta|1\rangle$

$= |\varphi\rangle_B$ ✓

$|\phi^-\rangle_{A_1A_2}$ Alice sends 11 $\Rightarrow$ Bob applies $Z_B X_B(\beta|0\rangle-\alpha|1\rangle)$

$= Z_B(\beta|1\rangle-\alpha|0\rangle)$

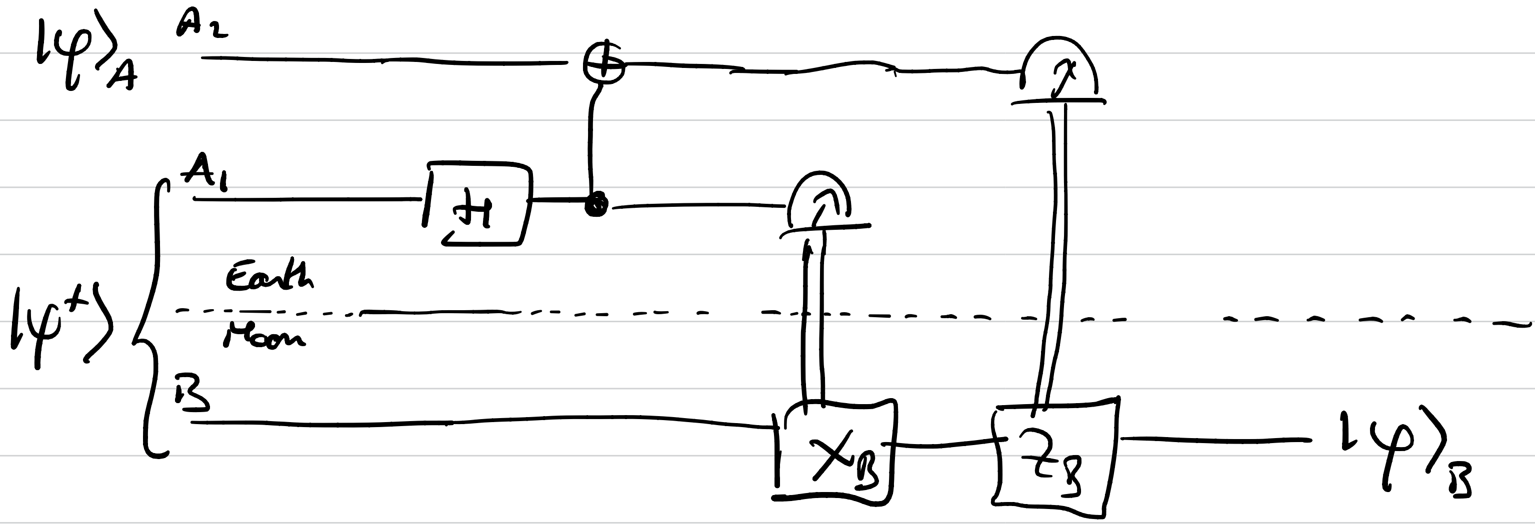
$= -\beta|1\rangle-\alpha|0\rangle$

$= -(\alpha|0\rangle+\beta|1\rangle)$

$\equiv |\varphi\rangle_B$ ✓

Teleportation Circuit.

A possible circuit representation is



The protocol can also be summarized as

$$1 \text{ qubit} = 2 \text{ c-bits} + 1 \text{ e-bit}.$$